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Non-standard quantum *so*(3, 2) and its contractions

Francisco J Herranz†

Departamento de Física, E U Politécnica, Universidad de Burgos, E-09006, Burgos, Spain

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Abstract. A full (triangular) quantum deformation of $so(3,2)$ is presented by considering this algebra as the conformal algebra of the $(2+1)$-dimensional Minkowskian spacetime. Non-relativistic contractions are analysed and used to obtain quantum Hopf structures for the conformal algebras of the $2+1$ Galilean and Carroll spacetimes. Relations between the latter and the null-plane quantum Poincaré algebra are studied.

1. Introduction

The Lie algebra $\mathcal{M}_3$ of the group of conformal transformations in the $(2+1)$-Minkowskian spacetime is a ten-dimensional Lie algebra isomorphic to $so(3,2)$. We consider the basis $\{J, P_0, P_i, K_i, C_0, C_i, D\}$ $(i=1,2)$ where J generates rotations, P_0 time translations, P_i space translations, K_i boosts, C_0 and C_i special conformal transformations, and D dilations. The Lie brackets of $\mathcal{M}_3$ are

$$\begin{array}{llll}
[J,K_i]=\epsilon_{ij}K_j & [J,P_i]=\epsilon_{ij}P_j & [J,C_i]=\epsilon_{ij}C_j & \\
[K_i,P_0]=P_i & [K_i,P_j]=\delta_{ij}P_0 & [K_1,K_2]=-J & \\
[K_i,C_0]=C_i & [K_i,C_j]=\delta_{ij}C_0 & [P_0,C_0]=D & \\
[P_0,C_i]=-K_i & [C_0,P_i]=-K_i & [P_i,C_j]=-\delta_{ij}D+\epsilon_{ij}J & \\
[D,P_\mu]=P_\mu & [D,C_\mu]=-C_\mu & [P_\mu,P_\nu]=0 & [C_\mu,C_\nu]=0 \\
[J,P_0]=0 & [J,C_0]=0 & [D,J]=0 & [D,K_i]=0
\end{array} \tag{1.1}$$

where ϵ_{ij} is antisymmetric with $\epsilon_{12}=1$, $\epsilon_{21}=-1$, $\epsilon_{ii}=0$, and from now on we assume that $\mu,\nu=0,1,2$ and $i,j=1,2$. The 2+1 Poincaré algebra, $\mathcal{P}(2+1)\equiv\langle J,P_0,P_i,K_i\rangle$, is a Lie subalgebra of $\mathcal{M}_3$; moreover, if we add the dilation generator D to $\mathcal{P}(2+1)$ we get the Weyl Lie subalgebra $\overline{\mathcal{P}}(2+1)$. Hence we have the sequence $\mathcal{P}(2+1)\subset\overline{\mathcal{P}}(2+1)\subset\mathcal{M}_3$.

The two non-relativistic limits of the Poincaré algebra $\mathcal{P}(2+1)$ are the Galilean $\mathcal{G}(2+1)$ and Carroll $\mathcal{C}(2+1)$ algebras which correspond, in this order, to a speed-space and a speed-time contraction of $\mathcal{P}(2+1)$ [1]. These contraction processes can be implemented at a conformal level in order to obtain the conformal algebras of the $2+1$ Galilean and Carroll spacetimes [2], here denoted $\mathcal{G}_3$ and $\mathcal{C}_3$, by applying the following mappings to the generators of $\mathcal{M}_3$:

$$\begin{array}{lllll}
\text{Speed-space contraction:} & J\to J & P_0\to P_0 & C_0\to C_0 & D\to D \\
\mathcal{M}_3\to\mathcal{G}_3 & P_i\to\varepsilon P_i & K_i\to\varepsilon K_i & C_i\to\varepsilon C_i &
\end{array} \tag{1.2}$$

$$\begin{array}{lllll}
\text{Speed-time contraction:} & J\to J & P_i\to P_i & C_i\to C_i & D\to D \\
\mathcal{M}_3\to\mathcal{C}_3 & P_0\to\varepsilon P_0 & K_i\to\varepsilon K_i & C_0\to\varepsilon C_0. &
\end{array} \tag{1.3}$$

† E-mail address: fteorica@cpd.uva.es

Once these transformations have been performed on the Lie brackets (1.1) we get after the limit $\varepsilon \to 0$ the commutation relations of $\mathcal{G}_3$ and $\mathcal{C}_3$. For the Galilean case, the non-vanishing commutators are:

$$\begin{array}{llll} [J, K_i] = \epsilon_{ij} K_j & [J, P_i] = \epsilon_{ij} P_j & [J, C_i] = \epsilon_{ij} C_j & \\ [K_i, P_0] = P_i & [K_i, C_0] = C_i & [P_0, C_0] = D & \\ [P_0, C_i] = -K_i & [C_0, P_i] = -K_i & [D, P_\mu] = P_\mu & [D, C_\mu] = -C_\mu . \end{array} \tag{1.4}$$

The conformal Galilean Lie algebra $\mathcal{G}_3$ is isomorphic to $t_6(so(2) \oplus so(2,1))$ (the structure of this type of algebra is described in [3, 4]). We also have a sequence of subalgebras $\mathcal{G}(2+1) \subset \overline{\mathcal{G}}(2+1) \subset \mathcal{G}_3$, where $\overline{\mathcal{G}}(2+1)$ is the $2+1$ Galilean algebra with dilation.

Likewise, we obtain the conformal Carroll algebra $\mathcal{C}_3$ with non-zero Lie brackets given by:

$$\begin{array}{llll} [J, K_i] = \epsilon_{ij} K_j & [J, P_i] = \epsilon_{ij} P_j & [J, C_i] = \epsilon_{ij} C_j & \\ [K_i, P_i] = P_0 & [K_i, C_i] = C_0 & [P_0, C_i] = -K_i & [C_0, P_i] = -K_i \\ [D, P_\mu] = P_\mu & [D, C_\mu] = -C_\mu & [P_i, C_j] = -\delta_{ij} D + \epsilon_{ij} J . & \end{array} \tag{1.5}$$

The embedding $\mathcal{C}(2+1) \subset \overline{\mathcal{C}}(2+1) \subset \mathcal{C}_3$ is easily verified ($\overline{\mathcal{C}}(2+1)$ means the Carroll Weyl subalgebra). The conformal algebra $\mathcal{C}_3$ turns out to be isomorphic to the 3+1 Poincaré algebra $iso(3,1)$. Recall that, in general, kinematical symmetries in $N+1$ dimensions can been seen as conformal symmetries in N dimensions [2].

Non-standard quantum deformations for these conformal algebras have been already obtained for the $1+1$ case [5, 6] being inspired in the well known non-standard or Jordanian quantum $sl(2,\mathbb{R})$ algebra [7–9]. Their underlying Lie bialgebras come from classical r-matrices which satisfy the classical Yang–Baxter equation. The results so obtained show that non-standard deformations are naturally adapted to a conformal basis, although for the particular case of the quantum Poincaré algebra an alternative interpretation has been considered in a null-plane framework [10].

An analysis of non-standard conformal Lie bialgebras for higher dimensions can be found in [11] where the deformation parameters are interpreted as fundamental mass parameters. However, to our knowledge, no explicit non-standard quantum Hopf structure for the conformal algebra further the $1+1$ case ($so(2,2)$) has been given yet. In this paper we solve this problem for a precise non-standard quantum deformation of $\mathcal{M}_3$. To begin with we consider in section 2 the $2+1$ conformal Lie bialgebra which generalizes that introduced in [5, 6], and we study its possible non-relativistic Lie bialgebra contractions. It is shown that there is a unique possible (coboundary) contraction for each conformal bialgebra $\mathcal{G}_3$ and $\mathcal{C}_3$. In section 3 the quantum Hopf structure of $\mathcal{M}_3$ is introduced and those corresponding to $\mathcal{G}_3$ and $\mathcal{C}_3$ are obtained via contraction in section 4. All of them have as Hopf subalgebra the corresponding kinematical algebra together with the dilation generator, but not the kinematical algebra itself (a feature already pointed out in [12]); hence only the Weyl subalgebra is promoted to a Hopf subalgebra. The quantum conformal Carroll algebra is related to the $3+1$ null-plane quantum Poincaré algebra; this fact allows us to obtain its universal R-matrix from the results given in [13].

2. Conformal Lie bialgebras

The classical r-matrices underlying the non-standard quantum deformations of $sl(2,\mathbb{R}) \equiv \langle P_0, C_0, D\rangle$ and $so(2,2) \equiv \langle P_0, P_1, K_1, C_0, C_1, D\rangle$ [5, 6] can be written as

$$r = z D \wedge P_0 \qquad r = z(D \wedge P_0 + K_1 \wedge P_1) \tag{2.1}$$

where z is the deformation parameter. The generalization of these expressions for $\mathcal{M}_3 \simeq so(3,2)$ reads

$$r = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + J \wedge P_2) \tag{2.2}$$

which fulfills the classical Yang–Baxter equation (the presence of the term $J \wedge P_2$ is essential for this purpose). The cocommutator of a generator X is obtained as $\delta(X) = [1 \otimes X + X \otimes 1, r]$, namely,

$$\begin{aligned}
&\delta(P_0) = 0 \qquad \delta(P_1) = 0\\
&\delta(P_2) = -zP_2 \wedge P_1 \qquad \delta(J) = -zJ \wedge P_1\\
&\delta(D) = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2 + J \wedge P_2)\\
&\delta(K_1) = z(K_1 \wedge P_0 + D \wedge P_1 - K_2 \wedge P_2 - J \wedge P_2)\\
&\delta(K_2) = z(K_2 \wedge P_0 + J \wedge P_0 + J \wedge P_1 + K_1 \wedge P_2 + D \wedge P_2)\\
&\delta(C_0) = z(C_0 \wedge P_0 - C_1 \wedge P_1 - C_2 \wedge P_2 - J \wedge K_2)\\
&\delta(C_1) = z(C_1 \wedge P_0 - C_0 \wedge P_1 - C_2 \wedge P_2 - J \wedge K_2)\\
&\delta(C_2) = z(C_2 \wedge P_0 + C_1 \wedge P_2 - C_0 \wedge P_2 + J \wedge K_1 + J \wedge D).
\end{aligned} \tag{2.3}$$

In order to analyse the possible non-relativistic contractions of this conformal Lie bialgebra one has to consider the Lie algebra transformations (1.2) and (1.3) together with a mapping on the deformation parameter, $z \to \varepsilon^{-n} z$, where n is any real number [14]. The result is that there exists a unique minimal value n_0 of n for each contraction from $\mathcal{M}_3$ to $\mathcal{G}_3$ and $\mathcal{C}_3$ in such way that the classical r-matrix and the cocommutators do not present divergencies:

$$\mathcal{M}_3 \to \mathcal{G}_3: \qquad z \to \varepsilon^{-2} z \qquad (n_0 = 2) \tag{2.4}$$

$$\mathcal{M}_3 \to \mathcal{C}_3: \qquad z \to \varepsilon^{-1} z \qquad (n_0 = 1). \tag{2.5}$$

For $n > n_0$ the contracted r-matrix and cocommutators go to zero and for $n < n_0$ they diverge.

The classical (non-standard) r-matrix and cocommutators of the conformal Galilean Lie bialgebra so obtained are given by

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2) \tag{2.6}$$

$$\begin{aligned}
&\delta(X) = 0 \qquad \text{for } X \in \{J, P_\mu, K_i, C_i\}\\
&\delta(D) = z(K_1 \wedge P_1 + K_2 \wedge P_2)\\
&\delta(C_0) = -z(C_1 \wedge P_1 + C_2 \wedge P_2)
\end{aligned} \tag{2.7}$$

and for the Carroll case we get

$$r = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2) \tag{2.8}$$

$$\begin{aligned}
&\delta(X) = 0 \qquad \text{for } X \in \{J, P_\mu\}\\
&\delta(Y) = zY \wedge P_0 \qquad \text{for } Y \in \{K_i, C_0\}\\
&\delta(D) = z(D \wedge P_0 + K_1 \wedge P_1 + K_2 \wedge P_2)\\
&\delta(C_1) = z(C_1 \wedge P_0 - C_0 \wedge P_1 - J \wedge K_2)\\
&\delta(C_2) = z(C_2 \wedge P_0 - C_0 \wedge P_2 + J \wedge K_1).
\end{aligned} \tag{2.9}$$

It is worth remarking that in each of the above Lie bialgebras the corresponding Weyl subalgebra $\{J, P_\mu, K_i, D\}$ is preserved at a bialgebra level. Note also that the cocommutator of D coincides with the classical r-matrix.

3. Quantum conformal Hopf algebra

We proceed to introduce the Jordanian quantum deformation of the conformal Minkowskian bialgebra, $U_z(\mathcal{M}_3)$, in two steps. First, we close the Hopf structure for the Weyl subalgebra, and second we complete the quantum deformation with the expressions involving the special conformal transformations. No direct procedure such as the deformation embedding method (applied, for instance, to the null-plane quantum Poincaré algebra [10]) seems to be useful now, so that we are forced to deduce formerly the coproduct Δ by solving the coassociativity condition

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta \tag{3.1}$$

and by taking into account the fact that the cocommutators (2.3) are related to the first order of Δ on z, $\Delta_{(1)}$, by means of

$$\delta = (\Delta_{(1)} - \sigma \circ \Delta_{(1)}) \qquad \text{where } \sigma(a \otimes b) = b \otimes a. \tag{3.2}$$

Thereafter, the deformed commutation rules follow by imposing the coproduct to be an algebra homomorphism, that is, $\Delta([X, Y]) = [\Delta(X), \Delta(Y)]$.

In the following we write down the coproduct and the commutation relations for $U_z(\mathcal{M}_3)$; the counit is trivial and the antipode can be easily derived from these results so we omit them.

(a) Weyl Hopf subalgebra $U_z(\overline{\mathcal{P}}(2+1))$.

$$\begin{aligned}
&\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1 \qquad \Delta(P_1) = 1 \otimes P_1 + P_1 \otimes 1 \\
&\Delta(P_2) = 1 \otimes P_2 + P_2 \otimes \mathrm{e}^{-zP_1} \qquad \Delta(J) = 1 \otimes J + J \otimes \mathrm{e}^{-zP_1} \\
&\Delta(D) = 1 \otimes D + D \otimes \mathrm{e}^{zP_0} \cosh zP_1 + K_1 \otimes \mathrm{e}^{zP_0} \sinh zP_1 \\
&\qquad\qquad + z(J + K_2) \otimes \mathrm{e}^{zP_0} P_2 + \tfrac{1}{2} z^2 (K_1 + D) \otimes \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \\
&\Delta(K_1) = 1 \otimes K_1 + K_1 \otimes \mathrm{e}^{zP_0} \cosh zP_1 + D \otimes \mathrm{e}^{zP_0} \sinh zP_1 \\
&\qquad\qquad - z(J + K_2) \otimes \mathrm{e}^{zP_0} P_2 - \tfrac{1}{2} z^2 (K_1 + D) \otimes \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \\
&\Delta(K_2) = 1 \otimes K_2 + (J + K_2) \otimes \mathrm{e}^{zP_0} + z(K_1 + D) \otimes \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2 - J \otimes \mathrm{e}^{-zP_1}
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&[J, K_i] = \epsilon_{ij} K_j \qquad [J, P_1] = P_2 \qquad [J, P_2] = \frac{1}{2z}(\mathrm{e}^{-2zP_1} - 1) - \frac{z}{2} P_2^2 \\
&[K_1, P_0] = \frac{1}{z} \mathrm{e}^{zP_0} \sinh zP_1 - \frac{z}{2} \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \qquad [K_2, P_0] = \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2 \\
&[K_1, P_1] = \frac{1}{z}(\mathrm{e}^{zP_0} \cosh zP_1 - 1) - \frac{z}{2} \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \qquad [K_1, P_2] = (1 - \mathrm{e}^{zP_0}\, \mathrm{e}^{-zP_1}) P_2 \\
&[K_2, P_2] = \frac{1}{z} \mathrm{e}^{-zP_1}(\mathrm{e}^{zP_0} - \cosh zP_1) + \frac{z}{2} P_2^2 \qquad [K_2, P_1] = (\mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} - 1) P_2 \\
&[K_1, K_2] = -J \qquad [D, P_0] = \frac{1}{z}(\mathrm{e}^{zP_0} \cosh zP_1 - 1) + \frac{z}{2} \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \\
&[D, P_1] = \frac{1}{z} \mathrm{e}^{zP_0} \sinh zP_1 + \frac{z}{2} \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2 \qquad [D, P_2] = \mathrm{e}^{zP_0}\, \mathrm{e}^{-zP_1} P_2 \\
&[P_\mu, P_\nu] = 0 \qquad [J, P_0] = 0 \qquad [D, J] = 0 \qquad [D, K_i] = 0.
\end{aligned} \tag{3.4}$$

(b) Special conformal transformations.

$$\begin{aligned}
\Delta(C_0) = {} & 1 \otimes C_0 + C_0 \otimes \mathrm{e}^{zP_0} \cosh zP_1 - C_1 \otimes \mathrm{e}^{zP_0} \sinh zP_1 - zC_2 \otimes \mathrm{e}^{zP_0} P_2 \\
& + z(J + K_2) \otimes \mathrm{e}^{zP_0} J + z^2 (K_1 + D) \otimes \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \\
& - \tfrac{1}{2} z^2 (C_1 - C_0) \otimes \mathrm{e}^{zP_0}\, \mathrm{e}^{zP_1} P_2^2
\end{aligned}$$

$$
\begin{aligned}
\Delta(C_1) = {} & 1 \otimes C_1 + C_1 \otimes \mathrm{e}^{zP_0} \cosh zP_1 - C_0 \otimes \mathrm{e}^{zP_0} \sinh zP_1 - zC_2 \otimes \mathrm{e}^{zP_0} P_2 \\
& + z(J + K_2) \otimes \mathrm{e}^{zP_0} J + z^2(K_1 + D) \otimes \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \\
& - \tfrac{1}{2} z^2 (C_1 - C_0) \otimes \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2^2 \\
\Delta(C_2) = {} & 1 \otimes C_2 + C_2 \otimes \mathrm{e}^{zP_0} + z(C_1 - C_0) \otimes \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 - z(K_1 + D) \otimes \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} J
\end{aligned}
\qquad (3.5)
$$

$$
\begin{aligned}
& [J, C_0] = -zK_1 J + \tfrac{1}{2} z J \qquad [J, C_1] = C_2 + zDJ \qquad [J, C_2] = -C_1 \\
& [K_1, C_0] = C_1 - \tfrac{1}{2} z(K_1 + D) + zK_1 D - z(J + K_2)^2 \\
& [K_2, C_0] = C_2 + \tfrac{1}{2} zK_2 + zK_1 J + z(K_1 + D)(J + K_2) \\
& [K_1, C_1] = C_0 - \tfrac{1}{2} z(K_1 + D) - \tfrac{1}{2} z(K_1^2 + D^2) - \tfrac{1}{2} z(J + K_2)^2 \\
& [K_2, C_2] = C_0 - \tfrac{1}{2} z(K_1 + D) - \tfrac{1}{2} z(K_1 + D)^2 - \tfrac{1}{2} z(J + K_2)^2 \\
& [K_1, C_2] = z(J + K_2)D \qquad [K_2, C_1] = -zDJ \qquad [P_2, C_2] = -D \\
& [P_0, C_0] = D - z\,\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \qquad [P_1, C_1] = -D - z\,\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \\
& [C_1, P_0] = K_1 + z\,\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \qquad [C_2, P_0] = K_2 - (\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} - 1)J \\
& [P_1, C_0] = K_1 - z\,\mathrm{e}^{zP_0} \mathrm{e}^{zP_1} P_2 J \qquad [P_2, C_1] = -\mathrm{e}^{zP_0} \mathrm{e}^{-zP_1} J + zDP_2 \\
& [P_2, C_0] = K_2 - zK_1 P_2 + \tfrac{1}{2} zP_2 - (\mathrm{e}^{zP_0} \mathrm{e}^{-zP_1} - 1)J \qquad [P_1, C_2] = \mathrm{e}^{zP_0} \mathrm{e}^{zP_1} J \\
& [D, C_0] = -C_0 + \tfrac{1}{2} z(K_1^2 + D^2) + \tfrac{1}{2} z(J + K_2)^2 \\
& [D, C_1] = -C_1 - zK_1 D \qquad [D, C_2] = -C_2 - z(J + K_2)D \\
& [C_1, C_2] = zC_2 - z(J + K_2)C_1 + z(K_1 + D)C_2 \\
& [C_0, C_1] = \tfrac{1}{2} z(C_1 - C_0) + z(J + K_2)C_2 \qquad [C_0, C_2] = -z(J + K_2)C_1 + \tfrac{1}{2} zC_2.
\end{aligned}
\qquad (3.6)
$$

Recall that the Drinfel'd–Jimbo q-deformation of $so(3,2)$ introduced in [15] was performed in a kinematical basis (as the algebra of the motion group of the $3+1$ anti-de Sitter spacetime) and the two primitive generators were a rotation and the time translation. Now the primitive generators are again two (the rank of the algebra): the time translation P_0 and a space translation P_1. On the other hand, the symmetry between P_μ and C_μ is broken in the quantum case (compare with (1.1)); for instance, all P_μ commute among themselves but the C_μ do not.

4. Quantum contractions

The contraction $U_z(\mathcal{M}_3) \to U_z(\mathcal{G}_3)$ is carried out by applying the transformations (1.2) and (2.4) to the results presented in the previous section. Once the limit $\varepsilon \to 0$ is taken, the resultant expressions are rather simplified. The coproduct and deformed commutation relations of the non-standard quantum conformal Galilean algebra $U_z(\mathcal{G}_3)$ are

$$
\begin{aligned}
& \Delta(X) = 1 \otimes X + X \otimes 1 \qquad \text{for } X \in \{J, P_\mu, K_i, C_i\} \\
& \Delta(D) = 1 \otimes D + D \otimes 1 + zK_1 \otimes P_1 + zK_2 \otimes P_2 \\
& \Delta(C_0) = 1 \otimes C_0 + C_0 \otimes 1 - zC_1 \otimes P_1 - zC_2 \otimes P_2
\end{aligned}
\qquad (4.1)
$$

$$
[D, P_0] = P_0 + \tfrac{1}{2} z(P_1^2 + P_2^2) \qquad [D, C_0] = -C_0 + \tfrac{1}{2} z(K_1^2 + K_2^2). \qquad (4.2)
$$

The remaining commutators are non-deformed and given by (1.4). On the other hand, the element

$$
\begin{aligned}
\mathcal{R} &= \exp\{r\} = \exp\{z(K_1 \wedge P_1 + K_2 \wedge P_2)\} \\
&= \exp\{-zP_2 \otimes K_2\} \exp\{-zP_1 \otimes K_1\} \exp\{zK_1 \otimes P_1\} \exp\{zK_2 \otimes P_2\}
\end{aligned}
\qquad (4.3)
$$

is a trivial solution of the quantum Yang–Baxter equation since the four generators involved commute. Furthermore, it is easy to check that the property

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X) \tag{4.4}$$

is satisfied for any $X \in U_z(\mathcal{G}_3)$. Then $\mathcal{R}$ is a quantum universal R-matrix for $U_z(\mathcal{G}_3)$.

Similarly, the contraction $U_z(\mathcal{M}_3) \to U_z(\mathcal{C}_3)$ is provided by the mappings (1.3) and (2.5) applied on the conformal Hopf algebra $U_z(\mathcal{M}_3)$ together with the limit $\varepsilon \to 0$. The coproduct and non-vanishing commutation relations of the quantum conformal Carroll algebra $U_z(\mathcal{C}_3)$ are given as follows:

$$\begin{aligned}
&\Delta(X) = 1 \otimes X + X \otimes 1 \qquad \text{for } X \in \{J, P_\mu\} \\
&\Delta(Y) = 1 \otimes Y + Y \otimes \mathrm{e}^{zP_0} \qquad \text{for } Y \in \{K_i, C_0\} \\
&\Delta(D) = 1 \otimes D + D \otimes \mathrm{e}^{zP_0} + zK_1 \otimes \mathrm{e}^{zP_0} P_1 + zK_2 \otimes \mathrm{e}^{zP_0} P_2 \\
&\Delta(C_1) = 1 \otimes C_1 + C_1 \otimes \mathrm{e}^{zP_0} - zC_0 \otimes \mathrm{e}^{zP_0} P_1 + zK_2 \otimes \mathrm{e}^{zP_0} J \\
&\Delta(C_2) = 1 \otimes C_2 + C_2 \otimes \mathrm{e}^{zP_0} - zC_0 \otimes \mathrm{e}^{zP_0} P_2 - zK_1 \otimes \mathrm{e}^{zP_0} J
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&[J, K_i] = \epsilon_{ij} K_j \qquad [J, P_i] = \epsilon_{ij} P_j \qquad [J, C_i] = \epsilon_{ij} C_j \\
&[K_i, P_i] = \frac{1}{z}(\mathrm{e}^{zP_0} - 1) \qquad [K_i, C_i] = C_0 - \frac{z}{2}(K_1^2 + K_2^2) \qquad [P_0, C_i] = -K_i \\
&[C_0, P_i] = -K_i \qquad [D, P_0] = \frac{1}{z}(\mathrm{e}^{zP_0} - 1) \qquad [D, P_i] = \mathrm{e}^{zP_0} P_i \\
&[D, C_0] = -C_0 + \frac{z}{2}(K_1^2 + K_2^2) \qquad [D, C_i] = -C_i - zK_i D \\
&[P_i, C_j] = -\delta_{ij} D + \epsilon_{ij}\, \mathrm{e}^{zP_0} J \qquad [C_1, C_2] = z(K_1C_2 - K_2C_1).
\end{aligned} \tag{4.6}$$

It is rather remarkable that $U_z(\mathcal{C}_3)$ can be shown to be isomorphic to the null-plane quantum Poincaré algebra [10] in the basis used in [13] by means of

$$\begin{aligned}
&P'_+ = P_0 \qquad P'_- = -C_0 \qquad P'_i = K_i \qquad J'_3 = -J \\
&K'_3 = D \qquad E'_i = -P_i \qquad F'_i = C_i \qquad z' = z/2
\end{aligned} \tag{4.7}$$

where the primed generators and deformation parameter correspond to the null-plane quantum Poincaré algebra. As a straightforward consequence the universal R-matrix for $U_z(\mathcal{C}_3)$ (satisfying the quantum Yang–Baxter equation and relation (4.4)) reads

$$\begin{aligned}
\mathcal{R} = {} & \exp\{-zP_2 \otimes K_2\} \exp\{-zP_1 \otimes K_1\} \exp\{-zP_0 \otimes D\} \\
& \times \exp\{zD \otimes P_0\} \exp\{zK_1 \otimes P_1\} \exp\{zK_2 \otimes P_2\}.
\end{aligned} \tag{4.8}$$

5. Concluding remarks

Summarizing, we have obtained a new quantum deformation of $so(3,2)$ and we have related three non-standard quantum conformal algebras via contraction processes, all of them containing the corresponding Weyl subalgebra as a Hopf subalgebra:

$$U_z(\overline{\mathcal{G}}(2+1)) \subset U_z(\mathcal{G}_3) \longleftarrow U_z(\overline{\mathcal{P}}(2+1)) \subset U_z(\mathcal{M}_3) \longrightarrow U_z(\overline{\mathcal{C}}(2+1)) \subset U_z(\mathcal{C}_3). \tag{5.1}$$

For the contracted quantum algebras the universal R-matrices have been given in a factorized form. The expression of the R-matrix associated to $U_z(\mathcal{M}_3)$ remains as an open problem.

We would like to note that we could have written the classical r-matrix for $so(2,2)$ (2.1) as

$$r = z(D \wedge P_1 + K_1 \wedge P_0). \tag{5.2}$$

Indeed, this was exactly the expression chosen in [5, 6]. Its generalization to $so(3,2)$ would be

$$r = z(D \wedge P_2 + K_1 \wedge P_1 + K_2 \wedge P_0 + J \wedge P_1). \tag{5.3}$$

From a mathematical point of view, the corresponding quantum deformation is equivalent to the one just studied via a simple redefinition of the generators. However, both deformations exhibit different physical features which are stronger when contractions are carried out. More explicitly, the quantum conformal Galilean and Carroll algebras coming from (5.3) are no longer equivalent to those above obtained. For both of them the transformation of z would be $z \to \varepsilon^{-2} z$ $(n_0 = 2)$ leading to the following classical r-matrices:

$$\mathcal{G}_3\colon r = zK_1 \wedge P_1 \qquad \mathcal{C}_3\colon r = zK_2 \wedge P_0. \tag{5.4}$$

Therefore, the latter could not be related to the null-plane quantum Poincaré algebra. The analysis of these and further possibilities will be presented elsewhere.

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